

Quantum Implementation of an LTI System with the Minimal Number of Additional Quantum Noise Inputs.

Shanon L. Vuglar, Ian R. Petersen

Abstract—Physical Realizability addresses the question of whether it is possible to implement a given LTI system as a quantum system. It is in general not true that a given synthesized quantum controller described by a set of stochastic differential equations is equivalent to some physically meaningful quantum system. However, if additional quantum noises are permitted in the implementation it is always possible to implement an arbitrary LTI system as a quantum system. In this paper we give an expression for the exact number of noises required to implement a given LTI system as a quantum system. Furthermore, we focus our attention on proving that this is a minimum, that is, it is not possible to implement the system as a quantum system with a smaller number of additional quantum noises.

I. INTRODUCTION

As the collective research effort into applications that rely on quantum effects intensifies, the topic of quantum control becomes increasingly relevant [1]–[9]. Such applications include quantum computing, quantum control and precision metrology. Quantum applications tend to be at the cutting edge of modern technology and have the potential to revolutionize the world we live in. However, their viability is dependent, at least in part, on quantum control. As an example, consider the recent results presented in [10]. Here, experimental success in maintaining a qubit in an oscillating superposition state through the use of feedback control is highly relevant to the field of quantum computing.

Of particular interest is the sub-field of coherent quantum control [1], [2], [5], [6]. By coherent quantum control, we mean that both the plant and controller are quantum systems coupled together via some type of quantum mechanism. An example could be an optical system coupled to a second optical system specifically designed to control some aspect of the first system. This type of setup has several advantages. Firstly, and perhaps most importantly, it avoids direct measurement of the quantum state of the system, thus avoiding the collapse of the quantum state that inevitably occurs during observation (measurement) and the associated loss of information. This is particularly relevant to quantum computing where we wish to manipulate quantum states. Secondly, it may be that by implementing the controller with the same type of system as the plant they will be of similar

Shanon L. Vuglar is with the School of Engineering and Information Technology, University of New South Wales at the Australian Defence Force Academy, Canberra ACT 2600, Australia. shanonvuglar@vuglar.com

Ian R. Petersen is with the School of Information Technology and Electrical Engineering, University of New South Wales at the Australian Defence Force Academy, Canberra ACT 2600, Australia. i.r.petersen@gmail.com

time scales, and there may be advantages in terms of speed of the control mechanism as compared to implementing the controller via some other method and coupling it to the quantum system via measurement and actuation.

When considering coherent quantum control the question of physical realizability naturally arises; given a controller that has been designed by some standard means (eg. LQG or H^∞ controller synthesis) is it possible to implement this controller as a quantum system? Unlike classical (i.e. not quantum) controllers which we may regard here as always being possible to implement at least approximately (but with arbitrary precision) via either analogue or digital electronics, it is in general not true that a given synthesized quantum controller described by a set of stochastic differential equations is equivalent to some physically meaningful quantum system. Several recent papers have addressed this issue of physical realizability [1], [9], [11]–[13], and of particular relevance is [1] in which the authors demonstrated that by incorporating additional quantum noises it is always possible to implement an arbitrary LTI system as a quantum system.

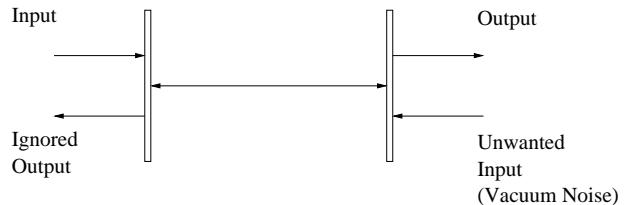


Fig. 1. Optical Cavity

To better understand what is meant by additional quantum noises consider the following rather simplistic example: suppose that in implementing a quantum controller as an optical system, the controller design calls for a laser to be passed through an optical cavity as shown in Figure 1. Here a naive approach would be to consider this device as having a single input and single output, however in fact there is a second input to the cavity: the mirror on the right which produces the output also causes the cavity to be coupled to a vacuum noise input; e.g. [14]. It turns out that to obtain correct results when modeling such a system it is imperative to take this additional vacuum noise source into account. It is in this way that additional quantum noise sources may inevitably enter the system when attempting to implement a controller as a quantum system; see also [4].

In general, additional noise sources when implementing a controller are undesirable and previous work [1], [9], [12],

[13], has gone some way to addressing the question of how many such noises are required in order to implement a given LTI system as a quantum system. In particular several previous results have provided upper and lower bounds on the number of additional quantum noises required as well as methods for determining when it is possible to implement a given quantum system with the additional noises equal to this lower bound. In this paper we provide a much stronger result; we give an expression for the exact number of noises required to implement an LTI system as a quantum system and focus our attention on proving that this is a minimum, that is, it is not possible to implement the system as a quantum system with a smaller number of additional quantum noises.

The remainder of the paper proceeds as follows. In Section II we describe the quantum system model used throughout this paper. In Section III we define what is meant by *physically realizable* and outline previous results. In Section IV we present our main result. An example and our conclusion follow in Sections V and VI respectively.

II. QUANTUM SYSTEM MODEL

As in [1], the linear quantum systems under consideration are assumed to be noncommutative stochastic systems described by quantum stochastic differential equations (QSDEs) of the form

$$\begin{aligned} dx(t) &= Ax(t) dt + [B_1 \quad B] \begin{bmatrix} dv(t) \\ du(t) \end{bmatrix}; \quad x(0) = x_0 \\ dy(t) &= Cx(t) dt + [D_1 \quad 0_{n_y \times n_u}] \begin{bmatrix} dv(t) \\ du(t) \end{bmatrix} \end{aligned} \quad (1)$$

where $x(t) = [x_1(t) \quad \dots \quad x_n(t)]^T$ is a column vector of n self-adjoint system variables. The noise $v(t) = [v_1(t) \quad \dots \quad v_{n_v}(t)]^T$ is a vector of noncommutative Wiener processes (in vacuum states) with Ito products $dv(t)dv^T(t) = F_v dt$ where F_v is non-negative Hermitian. $du(t)$ is a column vector of signals of the form $du(t) = \beta_u(t) dt + d\tilde{u}(t)$ where $\tilde{u}(t)$ is the noise part of $u(t)$ (with Ito products $d\tilde{u}(t)d\tilde{u}^T(t) = F_{\tilde{u}} dt$ where $F_{\tilde{u}}$ is non-negative Hermitian) and $\beta_u(t)$ is the adapted, self adjoint part of $u(t)$. $u(t)$ represents the input to the system. n, n_v, n_u and n_y are all assumed to be even (this is because in the quantum harmonic oscillator the system variables always occur as conjugate pairs, see [8]). A, B, C, B_1 and D_1 are appropriately dimensioned real matrices describing the dynamics of the system. For further details see [1].

For simplicity we restrict our attention to the case where $n_y = n_u$.

III. PHYSICAL REALIZABILITY

A. Definition

As in [1], [2], [9], [11], the concept of *physically realizable* means that the system dynamics described by (1) correspond to those of an open quantum harmonic oscillator.

Definition 1: By the *canonical commutation relations* we mean that the system variables x satisfy the commutation relations $[x_i(t), x_j(t)] = 2i\Theta_{ij}$ where Θ is a block diagonal matrix with each diagonal block equal to $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Definition 2: By *degenerate canonical commutation relations*, we mean that for the system (1), Θ as defined above is a block diagonal matrix with a zero matrix block followed by diagonal block's equal to J as above, along the diagonal.

The canonical commutation relations describe systems of a purely quantum nature whereas the degenerate canonical commutation relations describe hybrid systems consisting of both classical and quantum components.

Definition 3: (See [1, Definition 3.1]). The system described by (1) is an *open quantum harmonic oscillator* if Θ is canonical and there exist a quadratic Hamiltonian $H = \frac{1}{2}x(0)^T Rx(0)$, with a real, symmetric, $n \times n$ matrix R , and a coupling operator $L = \Lambda x(0)$, with complex-valued $\frac{1}{2}(n_v + n_u) \times n$ coupling matrix Λ , such that the matrices $A, [B_1 \quad B], C$ and $[D_1 \quad 0_{n_y \times n_u}]$ are given by:

$$A = 2\Theta (R + \mathfrak{Im}(\Lambda^\dagger \Lambda)); \quad (2)$$

$$[B_1 \quad B] = 2i\Theta [-\Lambda^\dagger \quad \Lambda^T] \Gamma; \quad (3)$$

$$C = P^T \begin{bmatrix} \Sigma_{n_y} & 0 \\ 0 & \Sigma_{n_y} \end{bmatrix} \begin{bmatrix} \Lambda + \Lambda^\# \\ -i\Lambda + i\Lambda^\# \end{bmatrix}; \quad (4)$$

$$[D_1 \quad 0_{n_y \times n_u}] = [I_{n_y \times n_y} \quad 0_{n_y \times (n_v + n_u - n_y)}]. \quad (5)$$

Here: $\Gamma_{(n_v+n_u) \times (n_v+n_u)} = P \text{diag}(M)$; $M = \frac{1}{2} \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$; $\Sigma_{n_y} = [I_{\frac{1}{2}n_y \times \frac{1}{2}n_y} \quad 0_{\frac{1}{2}n_y \times \frac{1}{2}(n_v+n_u-n_y)}]$; P is the appropriately dimensioned square permutation matrix such that $P \begin{bmatrix} a_1 & a_2 & \dots & a_{2m} \end{bmatrix} = \begin{bmatrix} a_1 & a_3 & \dots & a_{2m-1}a_2 & a_4 & \dots & a_{2m} \end{bmatrix}$ and $\text{diag}(M)$ is the appropriately dimensioned square block diagonal matrix with the matrix M occurring along the diagonal. (Note: dimensions of P and $\text{diag}(M)$ can always be determined from the context in which they appear.) $\mathfrak{Im}(\cdot)$ denotes the imaginary part of a matrix, $\#$ denotes the complex conjugate of a matrix, and † denotes the complex conjugate transpose of a matrix.

Definition 4: (See [1, Definition 3.3]). The system (1) is said to be physically realizable if one of the following holds:

- 1) Θ is canonical and (1) represents an open quantum harmonic oscillator.
- 2) Θ is degenerate canonical and there exists an augmentation which, after a suitable relabeling of the components, represents the dynamics of an open quantum harmonic oscillator [1].

Theorem 1: (see [1, Theorem 3.4]) The system (1) is physically realizable if and only if:

$$iA\Theta + i\Theta A^T + [B_1 \quad B] T_w [B_1 \quad B]^T = 0 \quad (6)$$

$$\begin{aligned} [B_1 \quad B] \begin{bmatrix} I \\ 0 \end{bmatrix} &= \Theta C^T P^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} P \\ &= \Theta C^T \text{diag}(J) \end{aligned} \quad (7)$$

and D_1 satisfies (5). Here

$$T_w = \frac{1}{2} \begin{bmatrix} F_v - F_v^T & 0 \\ 0 & F_{\tilde{u}} - F_{\tilde{u}}^T \end{bmatrix}.$$

In this paper we consider the problem of implementing a linear quantum system with a given transfer function as a fully quantum system and as such focus on the case where Θ is canonical.

B. Previous Results

In [1], it was demonstrated that by incorporating additional quantum noises, an arbitrary linear time invariant system could always be physically realized as a quantum system. In particular, the following lemma relating to the physical realizability of a purely quantum controller with canonical commutation relations was proved.

Lemma 1: (See [1, Lemma 5.6]). Suppose $F_u = I + iJ$, $F_v = I + iJ$, A, B and C are such that $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_u}$, $C \in \mathbb{R}^{n_y \times n}$, and $\Theta = \text{diag}(J)$ is canonical. Then there exists an even integer $n_v \geq n_y$ and matrix $B_1 \in \mathbb{R}^{n \times n_v}$ such that the system (1) is physically realizable.

Remark 1: By *minimal additional noises* we mean that $n_v = n_y$. It follows from [1, Theorem 3.4] that the number of outputs n_y is a lower bound on the number of additional noises n_v , necessary for physical realizability for a system described by a strictly proper transfer function.

In general, the incorporation of additional noises is undesirable, and in [9] the question of how many additional noises are required to implement a strictly proper LTI Quantum System was addressed. Drawing on the results contained in [1], in [9] an upper bound on the number of additional noises required was given. Further to this, a condition in terms of a certain non-standard Riccati equation was obtained for when a given transfer function (rather than a specific state space realization) could be physically realized using only a minimum number of additional quantum noises. The main results of [9] are reproduced here for the sake of completeness.

Theorem 2: (See [9, Theorem 1]). Consider an LTI system of the form (1) where A, B and C are given and the system commutation matrix Θ is canonical. There exists B_1 and D_1 such that the system is physically realizable with the number of quantum noises in \mathbf{dv} equal to $n_u + 2(n - n_\lambda)$ where n_λ is the multiplicity of the least (i.e. most negative) eigenvalue of the matrix $i(\Theta B \Theta B^T \Theta - \Theta A - A^T \Theta - C^T \Theta C)$.

In the special case that $n_\lambda = n$, it follows that $n_v = n_u$, which is the minimum number of quantum noises which need to be added for physically realizability.

IV. MAIN RESULT

Theorem 3: Consider an LTI system of the form (1) where A, B and C are given. There exists B_1 and D_1 such that the system is physically realizable with canonical commutation matrix Θ , and with the number of additional quantum noises n_v equal to $n_u + r$ where r is the rank of the matrix $(\Theta B \Theta B^T \Theta - \Theta A - A^T \Theta - C^T \Theta C)$. Conversely, suppose there exists B_1 and D_1 such that the system (1) where A, B and C are given is physically realizable with canonical commutation matrix Θ and the number of additional quantum noises equal to n_v . Then $n_v \geq n_u + r$ where r is the rank of the matrix $(\Theta B \Theta B^T \Theta - \Theta A - A^T \Theta - C^T \Theta C)$. That is, it is not possible to choose B_1 and D_1 such that the system is physically realizable and the number of additional quantum noises n_v is less than $n_u + r$.

Remark 2: Theorem 3 is stronger than Theorem 2 for two reasons: firstly, in general it will give a better result in

terms of the number of additional quantum noises sufficient for physical realizability, and secondly, Theorem 2 does not address the necessity of these noises whereas Theorem 3 does.

Remark 3: Theorem 2 and Theorem 3 are consistent with respect to the special case where the minimum number of additional quantum noises $n_v = n_u$ are sufficient (and necessary) for physical realizability. It will be shown in the proof that follows that $(\Theta B \Theta B^T \Theta - \Theta A - A^T \Theta - C^T \Theta C)$ has eigenvalues that occur in $+/ -$ pairs. As such, the case in Thereom 2 where $n_\lambda = n$, (that is all the eigenvalues are the same,) can only only occur when all the eigenvalues are identically zero which corresponds to the case in Theorem 3 where $r = 0$, (that is, $\Theta B \Theta B^T \Theta - \Theta A - A^T \Theta - C^T \Theta C = 0$).

Proof: The proof is structured as followed: we first show that $n_v = n_u + r$ additional noises are sufficient for physical realizability; we then show that $n_v \geq n_u + r$ additional noises are necessary.

In [1] a method was given to construct matrices R, Λ, B_1 and D_1 in (2) - (5) such that A, B and C were realized as required. Specifically the construction given in [1] is as follows:

$$\begin{aligned} D_1 &= [I_{n_y \times n_y} \ 0_{n_y \times (n_v - n_y)}]; \\ R &= -\frac{1}{4}(\Theta A + A^T \Theta^T); \\ B_1 &= [B_{1,1} \ B_{1,2}]; \\ \Lambda &= \left[\frac{1}{2} C^T P^T \begin{bmatrix} I \\ iI \end{bmatrix} \ \Lambda_{b1}^T \ \Lambda_{b2}^T \right]^T; \\ B_{1,1} &= \Theta C^T \text{diag}(J); \\ \Lambda_{b2} &= -i [I \ 0] P \text{diag}(M) B^T \Theta; \\ B_{1,2} &= 2i\Theta [-\Lambda_{b1}^\dagger \ \Lambda_{b1}^T] P \text{diag}(M) \end{aligned}$$

where $n_v \geq n_u + 2$. Here, Λ_{b1} is any complex $\frac{1}{2}(n_v - n_u) \times n$ matrix such that

$$\Lambda_{b1}^\dagger \Lambda_{b1} = \Xi_1 + i \begin{pmatrix} \frac{A^T \Theta^T - \Theta A}{4} \\ -\frac{1}{4} C^T P^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} P C \\ -\text{Im}(\Lambda_{b2}^\dagger \Lambda_{b2}) \end{pmatrix} \quad (8)$$

where Ξ_1 is any real symmetric $n \times n$ matrix such that $\Lambda_{b1}^\dagger \Lambda_{b1}$ is nonnegative definite.

Λ_{b1} is constructed as follows: a real symmetric $n \times n$ matrix Ξ_1 is first constructed such that the r.h.s. of (8) is nonnegative definite. Λ_{b1} is then constructed such that (8) holds.

Note: the construction of Λ_{b1} is of particular interest as Λ_{b1} has precisely $\frac{1}{2}(n_v - n_u)$ rows and thus determines the number of additional quantum noises required in this implementation.

Note: The construction above (as given in [1]) only allows for $n_v \geq n_u + 2$. However in the special case that the imaginary part of the r.h.s. of (8) is precisely zero, Ξ_1 can be chosen to be zero, Λ_{b1} and $B_{1,2}$ vanish, and $n_v = n_u$.

We now provide a method for choosing Ξ_1 and Λ_{b1} to obtain the result.

Note: It is desirable to construct Ξ_1 such that the $\Lambda_{b1}^\dagger \Lambda_{b1}$ is of minimum rank. This will allow Λ_{b1} to be constructed with minimum rows, thus the additional quantum noises required in this implementation, will be the minimum possible under this method of constructing R, Λ, B_1 and D_1 .

In [9] we show that (8) can be rewritten as

$$\Xi_2 = \Xi_1 + \frac{i}{4} \tilde{S}$$

where

$$\begin{aligned}\Xi_2 &= \Lambda_{b1}^\dagger \Lambda_{b1}; \\ \tilde{S} &= \Theta B \Theta B^T \Theta - A^T \Theta - \Theta A - C^T \Theta C.\end{aligned}$$

Note that the matrix \tilde{S} is real, skew symmetric. Thus $S = \frac{i}{4} \tilde{S}$ is hermitian, has real eigenvalues and is diagonalizable: $S = U^\dagger D U$ where D is diagonal and U is unitary.

We wish to find a real, symmetric Ξ_1 such that $\Xi_1 + S$ is positive semi-definite and of minimum rank. Let $\Xi_1 = U^\dagger |D| U$ where $|D|$ is the diagonal matrix with values equal to the absolute values of the corresponding entries in D . Then Ξ_1 is real, symmetric and $\Xi_2 = \Xi_1 + S \geq 0$. Further, Ξ_2 has rank equal to half that of S and this is the minimum rank possible for all allowed choices of Ξ_1 .

First we show that Ξ_1 so obtained is real and symmetric. Observe that $\Xi_1 = \Xi_1^\dagger; \Xi_1 \geq 0$. Also:

$$\Xi_1^2 = U^\dagger |D|^2 U = U^\dagger D^2 U = S^2$$

Here, S is purely imaginary, thus S^2 is real and $\Xi_1^2 \geq 0$ is also real, and therefore has a real square root. From the uniqueness of the positive semi-definite square root of a positive semi-definite matrix [15, Theorem 7.2.6], we conclude that Ξ_1 is real. Further, since Ξ_1 is Hermitian, Ξ_1 is symmetric.

We now show the rank condition. Observe the following about the eigenvalues of S : S is hermitian so its eigenvalues are real, thus the eigenvalues of \tilde{S} are purely imaginary, but since \tilde{S} is real its eigenvalues occur in complex conjugate pairs. Thus D is of the form:

$$\begin{aligned}D &= \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & \dots \\ 0 & -\lambda_1 & 0 & 0 & \dots \\ 0 & 0 & \lambda_2 & 0 & \dots \\ 0 & 0 & 0 & -\lambda_2 & \dots \\ \vdots & & & & \ddots \end{bmatrix}; \lambda_i \geq 0, \forall i; \\ |D| &= \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & \dots \\ 0 & \lambda_1 & 0 & 0 & \dots \\ 0 & 0 & \lambda_2 & 0 & \dots \\ 0 & 0 & 0 & \lambda_2 & \dots \\ \vdots & & & & \ddots \end{bmatrix}; \lambda_i \geq 0, \forall i; \\ |D| + D &= \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & \lambda_1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \lambda_2 & \dots \\ \vdots & & & & \ddots \end{bmatrix}; \lambda_i \geq 0, \forall i.\end{aligned}$$

From this, it can be seen that $|D| + D$ has rank, half that of D . Since

$$\begin{aligned}\Xi_2 &= \Xi_1 + S \\ &= U^\dagger |D| U + U^\dagger D U \\ &= U^\dagger (|D| + D) U,\end{aligned}$$

it follows that Ξ_2 has rank, half that of S .

Since S and \tilde{S} have the same rank, Ξ_2 has rank $\frac{r}{2}$ where r the rank of \tilde{S} . Since $\Xi_2 \geq 0$ has rank $\frac{r}{2}$, it is possible to construct Λ_{b1} with $\frac{r}{2}$ rows, such that $\Xi_2 = \Lambda_{b1}^\dagger \Lambda_{b1}$. Recall that, Λ_{b1} has precisely $\frac{1}{2}(n_v - n_u)$ rows, and we have $n_v = n_u + r$, that is, the system is physically realizable with the number of additional quantum noises n_v equal to $n_u + r$ where r is the rank of the matrix $\tilde{S} = (\Theta B \Theta B^T \Theta - \Theta A - A^T \Theta - C^T \Theta C)$.

We now consider the second part of the theorem and show that $n_v \geq n_u + r$ additional noises are necessary. To do so it is sufficient to show that the number of columns of $B_1 = [B_{1,1} \ B_{1,2}]$ is greater than or equal to $n_u + r$.

We consider the dimensions of $B_{1,1}$ first. From (7) we obtain

$$[B_{1,1} \ B_{1,2} \ B] \begin{bmatrix} I \\ 0 \end{bmatrix} = \Theta C^T \text{diag}(J).$$

By observing the dimensions of matrices on the l.h.s. this reduces to:

$$B_{1,1} = \Theta C^T \text{diag}(J),$$

Observe here that $B_{1,1}$ has precisely as many columns as C has rows, that is, $B_{1,1}$ has $n_y = n_u$ columns.

We next consider the dimensions of $B_{1,2}$. From (3) it can be shown that:

$$B = 2i\Theta \begin{bmatrix} -\Lambda_{b2}^\dagger & \Lambda_{b2}^T \end{bmatrix} \Gamma; \quad (9)$$

$$B_{1,1} = 2i\Theta \begin{bmatrix} -\Lambda_{b0}^\dagger & \Lambda_{b0}^T \end{bmatrix} \Gamma; \quad (10)$$

$$B_{1,2} = 2i\Theta \begin{bmatrix} -\Lambda_{b1}^\dagger & \Lambda_{b1}^T \end{bmatrix} \Gamma; \quad (11)$$

where

$$\Lambda = \begin{bmatrix} \Lambda_{b0} \\ \Lambda_{b1} \\ \Lambda_{b2} \end{bmatrix}.$$

That is, $B_{1,2}$ has twice the number of columns as Λ_{b1} has rows. We wish to show therefore, that Λ_{b1} has at least $\frac{r}{2}$ rows.

Consider,

$$\Im(\Lambda^\dagger \Lambda) = \Im(\Lambda_{b0}^\dagger \Lambda_{b0}) + \Im(\Lambda_{b1}^\dagger \Lambda_{b1}) + \Im(\Lambda_{b2}^\dagger \Lambda_{b2}).$$

That is,

$$\Im(\Lambda_{b1}^\dagger \Lambda_{b1}) = \Im(\Lambda^\dagger \Lambda) - \Im(\Lambda_{b0}^\dagger \Lambda_{b0}) - \Im(\Lambda_{b2}^\dagger \Lambda_{b2}).$$

Rearranging (2):

$$\frac{1}{2} \Theta^{-1} A = R + \Im(\Lambda^\dagger \Lambda),$$

where R and $\Im(\Lambda^\dagger \Lambda)$, are the symmetric and skew-symmetric parts respectively of the left hand side. From this it can be shown that

$$\Im(\Lambda^\dagger \Lambda) = -\frac{1}{4}(\Theta A + A^T \Theta).$$

Also, it is straightforward to verify that

$$\Im(\Lambda_{b0}^\dagger \Lambda_{b0}) = \frac{1}{4}(C^T \Theta C),$$

and

$$\Im(\Lambda_{b2}^\dagger \Lambda_{b2}) = -\frac{1}{4}(\Theta B \Theta B^T \Theta).$$

Substituting:

$$\begin{aligned} i \times \Im(\Lambda_{b1}^\dagger \Lambda_{b1}) \\ = \frac{i}{4} (\Theta B \Theta B^T \Theta - A^T \Theta - \Theta A - C^T \Theta C) \\ = \frac{i}{4} \tilde{S} = S. \end{aligned}$$

That is,

$$\Lambda_{b1}^\dagger \Lambda_{b1} = \Xi_1 + \frac{i}{4} \tilde{S},$$

where Ξ is the real part of $\Lambda_{b1}^\dagger \Lambda_{b1}$.

Consider the following fact [16, Fact 2.17.3]: Let $A, B \in \mathbb{R}^{n \times m}$, then

$$\text{rank}(A + jB) = \frac{1}{2} \text{rank} \begin{bmatrix} A & B \\ -B & A \end{bmatrix}.$$

From this observe that

$$\begin{aligned} \text{rank}(\Lambda_{b1}^\dagger \Lambda_{b1}) &= \text{rank} \left(\Xi_1 + i \frac{\tilde{S}}{4} \right) \\ &= \frac{1}{2} \text{rank} \begin{bmatrix} \Xi_1 & \frac{\tilde{S}}{4} \\ -\frac{\tilde{S}}{4} & \Xi_1 \end{bmatrix} \\ &\geq \frac{1}{2} \text{rank} \begin{bmatrix} \Xi_1 & \frac{\tilde{S}}{4} \end{bmatrix} \\ &\geq \frac{1}{2} \text{rank} \frac{\tilde{S}}{4} \end{aligned}$$

That is, independent of Ξ_1 ,

$$\text{rank}(\Lambda_{b1}^\dagger \Lambda_{b1}) \geq \frac{1}{2} \text{rank} S.$$

This in turn implies that Λ_{b1} has at least $\frac{r}{2}$ rows, where, as before, r is the rank of the matrix $\tilde{S} = (\Theta B \Theta B^T \Theta - \Theta A - A^T \Theta - C^T \Theta C)$. But from (10), $B_{1,2}$ has twice as many columns as Λ_{b1} has rows. That is, $B_{1,2}$ has at least r columns and hence B_1 has at least $n_u + r$ columns. As such, the number of additional noises n_v is greater than or equal to $n_u + r$. This concludes the proof of the theorem. \blacksquare

V. ILLUSTRATIVE EXAMPLE

In this section we consider a system from [1, Section VII,D], in which an example of classical-quantum controller synthesis is given, where the controller is implemented as a degenerate canonical controller with both classical and quantum degrees of freedom. Previously, in [12] it was shown that this system can be physically realized as a fully quantum system with $n_v = 8$ additional quantum noises. Here we apply our main result to show that this system can be implemented as a quantum system with only $n_v = 6$ additional quantum noises. Furthermore, it is not possible

to implement this system as quantum system with less than $n_v = 6$ additional quantum noises.

Consider a system of the form (1) with

$$\begin{aligned} A &= \begin{bmatrix} -1.3894 I_{2 \times 2} & -0.4472 I_{2 \times 2} \\ -0.2 I_{2 \times 2} & -0.25 I_{2 \times 2} \end{bmatrix}; \\ B &= \begin{bmatrix} -0.4472 I_{2 \times 2} \\ 0_{2 \times 2} \end{bmatrix}; \\ C &= \begin{bmatrix} -0.4472 I_{2 \times 2} & 0_{2 \times 2} \end{bmatrix}; \\ n &= 4; \quad n_u = 2; \quad \text{and} \quad n_y = 2. \end{aligned}$$

Applying *Theorem 3*:

$$\begin{aligned} \tilde{S} &= (\Theta B \Theta B^T \Theta - \Theta A - A^T \Theta - C^T \Theta C) \\ &= \begin{bmatrix} 0 & 2.3788 & 0 & 0.6472 \\ -2.3788 & 0 & -0.6472 & 0 \\ 0 & 0.6472 & 0 & 0.5 \\ -0.6472 & 0 & -0.5 & 0 \end{bmatrix} \end{aligned}$$

has rank $r = 4$, therefore the system is physically realizable with $n_v = n_u + r = 2 + 4 = 6$ additional quantum noises. Furthermore, it is not possible to physically realize this system with less than $n_v = 6$ additional quantum noises.

Remark: In [12] we show that if we are only concerned with implementing the transfer function described by $\{A, B, C\}$ in this example then it is possible to implement a different system $\{\tilde{A}, \tilde{B}, \tilde{C}\}$ with the same transfer function and only 2 additional noises.

VI. CONCLUSION

Physical Realizability is particularly pertinent to coherent quantum control where we wish to implement a given synthesized controller as a quantum system. By incorporating additional quantum noises in the implementation it is always possible to make a given LTI system physically realizable, however incorporating additional quantum noises is undesirable. In this paper we have given an expression for the number of additional quantum noises that are necessary to make a given LTI system physically realizable. Furthermore, we have shown that is not possible to make the given system physically realizable with a smaller number of additional quantum noises.

REFERENCES

- [1] M. R. James, H. I. Nurdin, and I. R. Petersen, “ H^∞ control of linear quantum stochastic systems,” *IEEE Transactions on Automatic Control*, vol. 53, no. 8, pp. 1787–1803, 2008.
- [2] A. I. Maalouf and I. R. Petersen, “Coherent H^∞ control for a class of linear complex quantum systems,” *IEEE Transactions on Automatic Control*, vol. 56, no. 2, pp. 309–319, 2011.
- [3] J. Gough and M. R. James, “The series product and its application to quantum feedforward and feedback networks,” *IEEE Transactions on Automatic Control*, vol. 54, no. 11, pp. 2530–2544, 2009.
- [4] H. I. Nurdin, M. R. James, and A. C. Doherty, “Network synthesis of linear dynamical quantum stochastic systems,” *SIAM Journal on Control and Optimization*, vol. 48, no. 4, pp. 2686–2718, 2009.
- [5] H. I. Nurdin, M. R. James, and I. R. Petersen, “Coherent quantum LQG control,” *Automatica*, vol. 45, no. 8, pp. 1837–1846, 2009.
- [6] H. Mabuchi, “Coherent-feedback quantum control with a dynamic compensator,” *Physical Review A*, vol. 78, p. 032323, 2008.
- [7] N. Yamamoto, “Robust observer for uncertain linear quantum systems,” *Phys. Rev. A*, vol. 74, pp. 032107–1 – 032107–10, 2006.
- [8] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control*. Cambridge University Press, 2010.

- [9] S. L. Vuglar and I. R. Petersen, "How many quantum noises need to be added to make an LTI system physically realizable?" in *Proceedings of the 2011 Australian Control Conference*, Melbourne, Australia, Novemeber 2011.
- [10] R. Vijay, C. Macklin, D. H. Slichter, S. J. Weber, K. W. Murch, R. Naik, A. N. Korotkov, and I. Siddiqi, "Stabilizing rabi oscillations in a superconducting qubit using quantum feedback," *Nature*, vol. 490, pp. 77–80, 2012.
- [11] A. I. Maalouf and I. R. Petersen, "Bounded real properties for a class of linear complex quantum systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 4, pp. 786 – 801, 2011.
- [12] S. Vuglar and I. Petersen, "A numerical condition for the physical realizability of a quantum linear system," in *Proceedings of the 20th International Symposium on Mathematical Theory of Networks and Systems, Melbourne, Australia*, 2012.
- [13] S. L. Vuglar and I. R. Petersen, "Singular perturbation approximations for general linear quantum systems," arXiv:1208.6155 [quant-ph].
- [14] C. Gardiner and P. Zoller, *Quantum Noise*. Berlin: Springer, 2000.
- [15] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, UK: Cambridge University Press, 1985.
- [16] D. S. Bernstein, *Matrix Mathematics: Theory, Facts, And Formulas with Application to Linear Systems Theory*. Princeton, New Jersey: Princeton University Press, 2005.